Journal of Global Optimization (2005) 33: 257–272
DOI 10.1007/s10898-004-1942-1

Generalized Nonlinear Lagrangian Formulation for Bounded Integer Programming

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(Received 26 November 2003; accepted in revised form 27 July 2004)

Abstract. Several nonlinear Lagrangian formulations have been recently proposed for bounded integer programming problems. While possessing an asymptotic strong duality property, these formulations offer a success guarantee for the identification of an optimal primal solution via a dual search. Investigating common features of nonlinear Lagrangian formulations in constructing a nonlinear support for nonconvex piecewise constant perturbation function, this paper proposes a generalized nonlinear Lagrangian formulation of which many existing nonlinear Lagrangian formulations become special cases.

Key words: duality gap, integer programming, Lagrangian relaxation, nonlinear integer programming, nonlinear Lagrangian formulation

1. Introduction

We consider in this paper the following singly constrained bounded integer programming problem:

$$\min\{f(x) : g(x) \leqslant 0, x \in X\}, \tag{1}$$

where f and $g : R^n \to R$ are continuous functions and X is a finite integer set. Note that an integer programming problem with multiple constraints can be always converted into an equivalent singly constrained problem by a nonlinear surrogate constraint method proposed in [4]. We thus concentrate our study in this paper on singly constrained problems for simplicity. Problem (1) is called the primal problem whose feasible region is given as follows:

$$S = \{x \in X : g(x) \leqslant 0\}. \tag{2}$$

Without loss of generality, we make the following two assumptions:

ASSUMPTION 1. $S \neq \emptyset$.

ASSUMPTION 2. $f(x) > 0$, *for all* $x \in X$.

Assumption 2 can be always satisfied by performing certain equivalent transformations on problem (1), for example, applying an exponential transformation on its objective function.

The concept of duality has been playing a significant role in optimization and Lagrangian methods [1–3, 8, 11] have been widely adopted in finding an optimal solution. Incorporating the constraint into the objective function by introducing a nonnegative Lagrangian multiplier, $\lambda \geqslant 0$, yields a Lagrangian relaxation:

$$\phi(\lambda) = \min_{x \in X} L(x, \lambda) = f(x) + \lambda g(x). \tag{3}$$

The Lagrangian dual is a maximization problem in λ,

$$\max_{\lambda \geq 0} \phi(\lambda). \tag{4}$$

However, the conventional Lagrangian method often fails to identify an optimal solution of the primal integer programming problem. As pointed out in [6], there are two critical situations that would prevent the Lagrangian method from succeeding in the dual search. In the first situation no optimal solution of (1) can be generated by solving (3) for any $\lambda \geqslant 0$. In the second situation the optimal solution of (3) with λ^* being a solution to the dual problem (4) is not an optimal solution to (1). One main reason behind these two kinds of failures is that there does not exist at the optimal point a linear support of the nonconvex piecewise-constant perturbation function.

Recent years have witnessed an extension from the traditional linear Lagrangian theory to an emerging nonlinear Lagrangian theory for integer programming [5, 6, 10, 12, 13]. The key concept in introducing nonlinear Lagrangian formulations is the construction of a nonlinear support of the perturbation function at the optimal point. While possessing an asymptotic strong duality property, the nonlinear Lagrangian formulations offer a success guarantee for the identification of an optimal primal solution via a dual search.

There could be many different forms of nonlinear supports. A natural question arisen is what are the common characteristics of various nonlinear Lagrangian formulations. Furthermore, what is a general form of nonlinear functions of the objective function f and the constraint function g that can serve as a nonlinear Lagrangian function? To be qualified as a nonlinear Lagrangian function, we require that the corresponding nonlinear Lagrangian formulation guarantee the identification of an optimal solution of the primal problem via a dual search.

In this paper, a generalized nonlinear Lagrangian formulation for bounded integer programming is presented. The proposed general formulation includes many specific nonlinear Lagrangian formulations proposed in the literature as its special cases. This generalized nonlinear Lagrangian formulation possesses an asymptotic strong duality property while it offers a success guarantee for the identification of a primal optimal solution via dual search. Another feature of this generalized nonlinear Lagrangian formulation is that no actual dual search is needed when the parameter in the formulation exceeds certain threshold.

The structure of the paper is as follows. Background and motivation of this research are given in Section 2. A new generalized Lagrangian formulation is introduced in Section 3 and its strong dual property is proved there too. The unimodality of the dual function is proved in Section 4. An attractive feature of the generalized Lagrangian formulation that no actual dual search is needed in searching for a primal optimal solution is derived in Section 5. Illustrative examples are provided in Section 6. Finally, a conclusion is drawn in Section 7.

2. Motivation

In this section, we investigate the perturbation function associated with problem (1). This study provides new insights into prominent features of nonlinear supports in nonlinear Lagrangian formulations.

A *perturbation function* of problem (1) with a single constraint is defined as follows for $z_1 \in R$:

$$\psi(z_1) = \min\{f(x) : g(x) \leqslant z_1, x \in X\}. \tag{5}$$

The domain of $\psi(\cdot)$ is

$$F = \{z_1 \in R : \text{ there exists } x \in X \text{ such that } g(x) \leqslant z_1\}.$$

Since X is finite, it is clear that the function $\psi(\cdot)$ is a nonincreasing piecewise-constant function of z_1 and is continuous from right. Let $z = (z_1, z_2)$ and define a set in R^2:

$$E = \{z : z_2 = \psi(z_1), z_1 \in F\}. \tag{6}$$

As defined, E is defining the function ψ over F. Geometrically, E is the lower envelope of the image of X in the (z_1, z_2) plane under the mapping $(g(x), f(x))$. Obviously, the image of the primal optimum point, P^*, is a point on E. To identify this optimal point on a nonincreasing piecewise-constant curve ψ, we need a class of functions whose nonlinear concave contours can support E at point P^*.

Sun and Li [10] constructed the following function in a nonlinear Lagrangian formulation,

$$C_p^1(z_1, z_2, \lambda) = \frac{1}{p}\ln\left[\frac{1}{2}(\exp(pz_2) + \exp(p\lambda z_1))\right], \tag{7}$$

where p is a parameter and $\lambda \geqslant 0$ is the Lagrangian multiplier of the nonlinear Lagrangian formulation. The domain of z_1 in contour $C_p^1(z_1, z_2, \lambda) = \alpha$ for any positive constant α and any $\lambda > 0$ is $(-\infty, (\ln(2)/p + \alpha)/\lambda)$. The slope at any point (z_1, z_2) on the contour $C_p^1(z_1, z_2, \lambda) = \alpha$ is

$$\frac{dz_2}{dz_1} = -\frac{\lambda}{2\exp(p(\alpha - \lambda z_1)) - 1}. \tag{8}$$

Formula (8) possesses following features:

$$\frac{dz_2}{dz_1} < 0, \quad \lambda > 0, \quad z_1 \in (-\infty, (\ln(2)/p + \alpha)/\lambda), \tag{9}$$

$$\frac{dz_2}{dz_1} \to 0, \quad p \to \infty, \quad z_1 \in (-\infty, \alpha/\lambda), \tag{10}$$

$$\frac{dz_2}{dz_1} \to -\infty, \quad \lambda \to \infty, \quad z_1 \in (0, (\ln(2)/p + \alpha)/\lambda). \tag{11}$$

Xu and Li [12] gave another function in their study of nonlinear Lagrangian theory,

$$C_p^2(z_1, z_2, \lambda) = z_2 + \frac{\exp(\lambda z_1)}{\lambda}, \quad \lambda > p > 0, \tag{12}$$

where p is a parameter and λ is the Lagrangian multiplier of the nonlinear Lagrangian formulation. The domain of z_1 in contour $C_p^2(z_1, z_2, \lambda) = \alpha$ for any positive constant α and any $\lambda > p > 0$ is $(-\infty, \ln(\lambda\alpha)/\lambda)$. The slope at any point (z_1, z_2) on the contour $C_p^2(z_1, z_2, \lambda) = \alpha$ is

$$\frac{dz_2}{dz_1} = -\exp(\lambda z_1). \tag{13}$$

Formula (13) also possesses the similar features as formula (8):

$$\frac{dz_2}{dz_1} < 0, \quad \lambda > 0, \quad z_1 \in \left(-\infty, \frac{\ln(\lambda\alpha)}{\lambda}\right), \tag{14}$$

$$\frac{dz_2}{dz_1} \to 0, \quad p \to \infty, \quad z_1 \in (-\infty, 0), \tag{15}$$

$$\frac{dz_2}{dz_1} \to -\infty, \quad \lambda \to \infty, \quad z_1 \in \left(0, \frac{\ln(\lambda\alpha)}{\lambda}\right). \tag{16}$$

We can observe some common properties in both nonlinear Lagrangian formulas (7) and (12). In view of (9) and (14), z_2 in both formulas is a strictly decreasing function of z_1 in its domain when $\lambda > 0$. Furthermore, (10), (11), (15) and (16) reveal that, if p is chosen large enough, the value of z_2 on either $C_p^1(z_1, z_2, \lambda) = \alpha$ or $C_p^2(z_1, z_2, \lambda) = \alpha$ would decrease very slowly when z_1 is negative, while, if λ is chosen large enough, it would decrease rapidly when z_1 is positive. Geometrically, when parameters p and λ are set large enough for negative and positive z_1, respectively, both contours of $C_p^1(z_1, z_2, \lambda) = \alpha$ and $C_p^2(z_1, z_2, \lambda) = \alpha$ approach a horizontal line when z_1 is negative and a vertical line when z_1 is positive, and possess a right angle at $z_1 = 0$. Such curves can be considered as an approximation of a shifted negative octant which can support any nonincreasing perturbation function, no matter it is nonconvex or not. Hence, such contours offer a nonlinear support to E, and ensure a unique support at the point P^*, where a linear support in terms of z_1 and z_2 may not exist.

The discussion above inspires us to explore what is the essence of the nonlinear Lagrangian formulations and to construct a general nonlinear Lagrangian function $C_p(z_1, z_2, \lambda)$ based on the features exhibited in nonlinear Lagrangian formulas (7) and (12).

Let the contour of a nonlinear Lagrangian function be $C_p(z_1, z_2, \lambda) = \alpha$. We have

$$\frac{dz_2}{dz_1} = -\frac{\partial C_p(z_1, z_2, \lambda)/\partial z_1}{\partial C_p(z_1, z_2, \lambda)/\partial z_2}. \tag{17}$$

In view of (9)–(12), (14)–(17), we can draw some conclusions as follows.

- A sufficient condition for (9) and (14) to hold is that $C_p(z_1, z_2, \lambda)$ is strictly increasing with respect to both z_1 and z_2 for any $\lambda > 0$.
- If $\partial C_p(z_1, z_2, \lambda)/\partial z_2$ has a strictly positive lower bound, then (10) and (15) imply that $\partial C_p(z_1, z_2, \lambda)/\partial z_1 \to 0$ for negative z_1 as $p \to \infty$. This means that $C_p(z_1, z_2, \lambda)$ becomes independent of z_1 for sufficiently large p.
- (11) and (16) are equivalent to

$$\frac{\partial C_p(z_1, z_2, \lambda)/\partial z_1}{\partial C_p(z_1, z_2, \lambda)/\partial z_2} \to \infty \quad \text{as} \quad \lambda \to +\infty \quad \text{for any } x \in X \setminus S.$$

If $\partial C_p(z_1, z_2, \lambda)/\partial z_2$ has a strictly positive lower bound, then it must hold that $\partial C_p(z_1, z_2, \lambda)/\partial z_1 \to +\infty$ as $\lambda \to +\infty$ for any positive z_1. This implies $C_p(z_1, z_2, \lambda) \to \infty$ as $\lambda \to +\infty$ for any positive z_1.

3. Generalized Lagrangian formulation

We propose in this section a general form for nonlinear Lagrangian formulation and prove its asymptotic strong duality property.

From the analysis in the last section, a generalized Lagrangian function (GLF) should satisfy the followings: (i) For any $x \in X \setminus S$, GLF tends to infinity as λ tends to infinity; and (ii) for any $x \in S$, GLF does not depend on $g(x)$ when parameter p is sufficiently large. If we let GLF converge to $f(x)$ as parameter p becomes sufficiently large, then the GLF will not depend on $g(x)$. Now we introduce the definition of GLF.

DEFINITION 1. A continuous function $L_p(g(x), f(x), \lambda)$ with parameters $p > 0$ and $\lambda > 0$ is called a generalized Lagrangian function (GLF) of problem (1) if it satisfies the following two conditions:

(i) For any $x \in S$, $L_p(g(x), f(x), \lambda) \to f(x)$ as $p \to \infty$.
(ii) For any $x \in X \setminus S$, $L_p(g(x), f(x), \lambda) \to +\infty$ as $\lambda \to \infty$.

In view of Definition 1, the conventional linear Lagrangian function $L(x, \lambda) = f(x) + \lambda g(x)$ is not a GLF, since the condition (i) of Definition 1 is unsatisfied, i.e., $L(x,\lambda) \not\to f(x)$ for any $x \in S$. Now we list some candidates of GLF. It is easy to conclude that these examples satisfy two conditions in Definition 1.

EXAMPLE 1. The logarithmic-exponential Lagrangian function of problem (1) defined in [10],

$$L_p(g(x), f(x), \lambda) = \frac{1}{p}\ln\left[\frac{1}{2}(\exp(pf(x)) + \exp(p\lambda g(x)))\right],$$

is a GLF, where $\lambda \geqslant 0$.

EXAMPLE 2. The exponential Lagrangian function defined in [12],

$$L_p(g(x), f(x), \lambda) = f(x) + \frac{1}{\lambda}\exp(\lambda g(x)), \ \lambda \geqslant p > 0,$$

is a GLF.

EXAMPLE 3. The logarithmic-exponential penalty function defined in [9],

$$L_p(g(x), f(x), \lambda) = f(x) + \frac{\lambda}{p}\ln[1 + \exp(\lambda g(x))],$$

is a GLF.

EXAMPLE 4. The following function is also a GLF:

$$L_p(g(x), f(x), \lambda) = [f(x)^p + \exp(p\lambda g(x))]^{1/p}.$$

We present some properties of a GLF in the following lemma without proof, since they are clear from the definition of $L_p(g(x), f(x), \lambda)$ in Definition 1.

LEMMA 1. (i) *For a given* $x \in S$ *and any* $\varepsilon > 0$, *there exists* a $p(x,\varepsilon) > 0$ *such that for* $p > p(x,\varepsilon)$,

$$f(x) - \varepsilon \leqslant L_p(g(x), f(x), \lambda) \leqslant f(x) + \varepsilon. \tag{18}$$

(ii) *For a given* $x \in X \setminus S$ *and any* $M > 0$, *there exists a* $\lambda(x, M) > 0$ *such that for* $\lambda > \lambda(x, M)$,

$$L_p(g(x), f(x), \lambda) \geqslant M. \tag{19}$$

The GLF-based Lagrangian relaxation problem associated with (1) is defined as

$$\phi_p(\lambda) := \min_{x \in X} L_p(g(x), f(x), \lambda). \tag{20}$$

Further, the GLF-based Lagrangian dual problem associated with (1) is defined as

$$D_p := \max_{\lambda \geq 0} \phi_p(\lambda). \tag{21}$$

Now we prove the asymptotic strong duality property of the generalized Lagrangian formulation given in (20) and (21). For simplicity, denote

$$f^* = \min_{x \in S} f(x).$$

From Assumption 2, we have $f^* > 0$.

THEOREM 1. (*Asymptotic Strong Duality Property*). *Suppose that $L_p(g(x), f(x), \lambda)$ is a GLF and D_p is defined by* (20) *and* (21). *Then*

$$\lim_{p \to \infty} D_p = f^*.$$

If $S = X$, then $\lim_{p\to\infty} D_p = \min_{x\in S} f(x)$ holds trivially by (20), (21) and part (i) of Lemma 1. Now suppose $X \setminus S \neq \emptyset$. Again from part (i) of Lemma 1, for any $\varepsilon > 0$ and sufficiently large p, we have

$$\begin{aligned} D_p &= \max_{\lambda \geq 0} \min_{x \in X} L_p(g(x), f(x), \lambda) \\ &\leqslant \max_{\lambda \geq 0} \min_{x \in S} L_p(g(x), f(x), \lambda) \\ &\leqslant \max_{\lambda \geq 0} \min_{x \in S} (f(x) + \varepsilon) \\ &= f^* + \varepsilon. \end{aligned} \tag{22}$$

Now we assert that for any sufficiently large $p > 0$, there exists a $\lambda > 0$ such that

$$\min_{x \in X \setminus S} L_p(g(x), f(x), \lambda) \geqslant \min_{x \in X} L_p(g(x), f(x), \lambda). \tag{23}$$

Suppose that, on the contrary, there exists no $\lambda > 0$ such that (23) holds. Then, for any $\lambda > 0$, we have

$$\begin{aligned} D_p &\geqslant \phi_p(\lambda) \\ &= \min\{\min_{x \in X\setminus S} L_p(g(x), f(x), \lambda), \min_{x \in S} L_p(g(x), f(x), \lambda)\} \\ &= \min_{x \in X \setminus S} L_p(g(x), f(x), \lambda). \end{aligned} \tag{24}$$

Let $M = f^* + 2\varepsilon$. From part (ii) of Lemma 1, $\forall x \in X\setminus S$, there exists a $\hat{\lambda} > 0$ such that $L_p(g(x), f(x), \hat{\lambda}) \geqslant f^* + 2\varepsilon$. Setting $\lambda = \hat{\lambda}$, we get from (24) that

$$D_p \geqslant \min_{x \in X\setminus S} L_p(g(x), f(x), \hat{\lambda}) \geqslant f^* + 2\varepsilon. \tag{25}$$

Equation (25) shows a contradiction to (22). Therefore, there must exist a $\lambda^* > 0$ such that (23) holds. In views of part (i) of Lemma 1 and (23), we have

$$\begin{aligned} D_p &\geqslant \phi_p(\lambda^*) \\ &= \min\{\min_{x \in X\setminus S} L_p(g(x), f(x), \lambda^*), \min_{x \in S} L_p(g(x), f(x), \lambda^*)\} \\ &= \min_{x \in S} L_p(g(x), f(x), \lambda^*) \\ &\geqslant f^* - \varepsilon. \end{aligned} \tag{26}$$

Combining (22) and (26) yields that for any $\varepsilon > 0$ and sufficiently large $p > 0$, we have $f^* - \varepsilon \leqslant D_p \leqslant f^* + \varepsilon$. This comes to the conclusion.

Theorem 1 reveals that the optimal value of the Lagrangian dual problem attains the optimal value of primal problem (1) when p approaches infinity. In implementation, we are more interested in achieving the primal optimality with a finite p. Once the parameter p exceeds a threshold, an optimal solution of primal problem (1) can be identified by the proposed generalized nonlinear Lagrangian formula. For convenience, the following two notations are introduced,

$$S^* = \{x \in S : f(x) = f^*\},$$

$$\delta = \min\{f(x) : x \in S \backslash S^*\} - f^*.$$

LEMMA 2. *There exists a* $p^* > 0$ *such that for any* $p > p^*$, *any optimum solution* x^* *of* (20) *satisfying* $x^* \in S$ *is an optimal solution of problem* (1).

In view of part (i) of Lemma 1 and Theorem 1, given $\varepsilon = \delta/4$, there exists p^* such that for any $p > p^*$,

$$f(x^*) - \varepsilon \leqslant L_p(g(x^*), f(x^*), \lambda) = \min_{x \in X} L_p(g(x), f(x), \lambda) \leqslant f^* + \varepsilon.$$

Hence

$$f(x^*) - f^* \leq 2\varepsilon = \frac{\delta}{2}.$$

This implies $x^* \in S^*$ by the definition of δ.

4. The unimodality of dual function

In this section, we continue to explore the properties of $L_p(g(x), f(x), \lambda)$ and the dual formulas in (20) and (21). Specifically, we will show the unimodality of the dual function. Notice that the dual function in the traditional linear Lagrangian framework is concave, thus possessing the unimodality. The dual function in nonlinear Lagrangian in general is not concave as witnessed in [10]. On the other hand, the property of the unimodality will also guarantee that a local maximum of the dual function is also a global maximum, thus facilitating the dual search.

From the analysis in Section 2, the monotonically increasing property of a nonlinear Lagrangian function with respect to both $f(x)$ and $g(x)$ is another desirable feature of nonlinear Lagrangian functions. Attaching this property to the definition of a GLF leads to the definition of a regular GLF.

DEFINITION 2. A GLF is called regular if it satisfies following additional three conditions:

(i′) For any $x \in X \backslash S$, $L_p(g(x), f(x), \lambda)$ is strictly increasing with respect to λ.

(ii′) For given $\lambda > 0$, $L_p(g(x), f(x), \lambda)$ is strictly increasing with respect to both $g(x)$ and $f(x)$.

(iii′) For any $x \in S$, $L_p(g(x), f(x), \lambda)$ is decreasing with respect to λ.

It is easy to verify that the nonlinear Lagrangian functions in Examples 1, 3 and 4 are all regular, while the nonlinear Lagrangian function in Example 2 is regular when parameter p exceeds a certain threshold.

Observe that the perturbation function $\psi(\cdot)$ defined by (5) is continuous from right and the domain F of $\psi(\cdot)$ is an interval. By the finiteness of X, the number of the discontinues points of $\psi(\cdot)$ is finite. Without loss of generality, list them as $\{a_1, \ldots, a_{K+L}\}$ with

$$a_1 < a_2 < \cdots < a_K \leqslant 0 < a_{K+1} < \cdots < a_{K+L}. \tag{27}$$

By Assumption 1, we have $K \geqslant 1$. Let $b_i = \psi(a_i)$, for $i = 1, \ldots, K+L$. Observe that the perturbation function is nonincreasing and its discontinues points are strictly decreasing. We have from (5) and Assumption 2 that

$$b_1 > b_2 > \cdots > b_K > b_{K+1} > \cdots > b_{K+L} > 0. \tag{28}$$

Let

$$S(E) = \{(a_i, b_i) \in R^2 : i = 1, \ldots, K+L\}. \tag{29}$$

It is clear that point $(z_1, z_2) \in S(E)$ if and only if $(z_1, z_2) \in E$, the lower envelope set defined by (6), and $(\hat{z}_1, z_2) \notin E$ for any $\hat{z}_1 < z_1$. Point (a_i, b_i) is associated with a feasible solution of problem (1) when $1 \leqslant i \leqslant K$ and with an infeasible solution of problem (1) when $K+1 \leqslant i \leqslant K+L$.

LEMMA 3. (1) *For any* $p > 0$, *if* x^* *is an optimum solution of* (20) *for a given* $\lambda > 0$, *then* $(g(x^*), f(x^*)) \in S(E)$.

(2) *There exists at least one optimum solution* $x^* \in S^*$ *such that* $(g(x^*), f(x^*)) \in S(E)$.

For the first part, suppose that $(g(x^*), f(x^*)) \in E$, but $(g(x^*), f(x^*)) \notin S(E)$. Then, by the definitions of E and $S(E)$ in (6) and (29), there exists an $\hat{x} \in X$ such that $f(\hat{x}) \leqslant f(x^*)$ and $g(\hat{x}) < g(x^*)$. Hence, by (ii′) of Definition 2, for any $p > 0$, we have $L_p(g(\hat{x}), f(\hat{x}), \lambda) < L_p(g(x^*), f(x^*), \lambda)$. This is a contradiction to the optimality of x^* in (20).

For the second part, it is clear that $(g(x), f(x)) \in E$ holds for any $x \in S^*$. Let $x^* \in \arg\{g(y) : y \in S^*\}$. For any $z_1 < g(x^*)$, we have $S^* \cap \{x : g(x) \leqslant z_1, x \in X\} = \phi$. So, $\psi(z_1) > f(x^*)$, which implies $(z_1, f(x^*)) \notin E$. Hence, $(g(x^*), f(x^*)) \in S(E)$. We complete the proof.

Let

$$l_p^i(\lambda) = L_p(a_i, b_i, \lambda), \quad i = 1, \ldots, K+L. \tag{30}$$

Then by Lemma 3, we have

$$\phi_p(\lambda) = \min_{x \in X} L_p(g(x), f(x), \lambda) = \min_{1 \leqslant i \leqslant K+L} l_p^i(\lambda). \tag{31}$$

The theorem below reveals that the function defined by (20) is a unimodal function when p is sufficiently large. Denote $I = I_1 \cup I_2$, where $I_1 = \{i|1 \leqslant i \leqslant K\}$ and $I_2 = \{i|K+1 \leqslant i \leqslant K+L\}$.

THEOREM 2. *Suppose that $L_p(g(x), f(x), \lambda)$ is a regular GLF. Then, for any $p > 0$, there exists a $\lambda^*(p) > 0$ such that the dual function $\phi_p(\lambda)$is monotonically increasing in $[0, \lambda^*(p)]$ and monotone decreasing in $[\lambda^*(p), \infty)$.*

First, we prove that for any $p > 0$, if there exists $\lambda_1 > 0$ such that $\phi_p(\lambda_1) = l_p^{i_1}(\lambda_1)$, where $i_1 \in I_1$, then for any $\lambda_2 > \lambda_1$, there must be $\phi_p(\lambda_2) = l_p^{i_2}(\lambda_2)$ satisfying $i_2 \in I_1$. Suppose, on the contrary, that there exists a $\lambda_2 > \lambda_1$ such that

$$\begin{aligned}\phi_p(\lambda_2) &= \min_{i \in I} l_p^i(\lambda_2) \\ &= \min\{\min_{i \in I_1} l_p^i(\lambda_2), \min_{i \in I_2} l_p^i(\lambda_2)\} \\ &= \min_{i \in I_2} l_p^i(\lambda_2) \\ &= l_p^{i_2}(\lambda_2), \quad i_2 \in I_2.\end{aligned}$$

Then for any $i \in I_1$,

$$\phi_p(\lambda_2) = l_p^{i_2}(\lambda_2) \leqslant l_p^i(\lambda_2). \tag{32}$$

From (iii′) of Definition 2, $L_p(g(x), f(x), \lambda)$ is decreasing about λ when $x \in S$. Hence, for given $i_1 \in I_1$, we have

$$l_p^{i_1}(\lambda_2) \leqslant l_p^{i_1}(\lambda_1). \tag{33}$$

Since $\phi_p^{i_1}(\lambda_1) = l_p^{i_1}(\lambda_1)$, where $i_1 \in I_1$, then for given $i_2 \in I_2$ in (32) we have

$$l_p^{i_1}(\lambda_1) \leqslant l_p^{i_2}(\lambda_1). \tag{34}$$

Combining (32)–(34), we obtain

$$l_p^{i_1}(\lambda_1) \leqslant l_p^{i_2}(\lambda_1) < l_p^{i_2}(\lambda_2) \leqslant l_p^{i_1}(\lambda_1), \tag{35}$$

where the second inequality holds from item (i') of Definition 2 and $\lambda_1 < \lambda_2$. This is a contradiction.

In the same way, we can also assert that for given $p > 0$, if there exists $\lambda_1 > 0$ such that $\phi_p(\lambda_1) = l_p^{i_1}(\lambda_1), i_1 \in I_2$, then for any $0 < \lambda_2 < \lambda_1$, there must be $\phi_p(\lambda_2) = l_p^{i_2}(\lambda_2), i_2 \in I_2$.

The conclusions above imply that there exists $\lambda^*(p) > 0$ such that for any $\lambda \in [0, \lambda^*(p)]$, $\phi_p(\lambda) = l_p^i(\lambda)$ with $i \in I_2$ and for any $\lambda \in [\lambda^*(p), \infty)$, $\phi_p(\lambda) = l_p^i(\lambda)$ with $i \in I_1$. Since the function $l_p^i(\lambda)$ corresponding to $i \in I_2$ is increasing by (i′) of Definition 2 and that corresponding to $i \in I_1$ is decreasing by (iii′) of Definition 2, the conclusion is true.

From Theorem 2, we can immediately obtain a corollary as follows.

COROLLARY 1. *For any* $p > 0$*, the dual problem (21) has a unique finite solution* $\lambda^*(p)$.

5. Primal optimum solution via Lagrangian relaxation

In this section, we focus on how to obtain a primal optimum solution of problem (1) by solving a Lagrangian relaxation problem. The theorem below reveals that no actual dual search is needed when p is large enough.

LEMMA 4. *Suppose that* $L_p(g(x), f(x), \lambda)$ *is a regular GLF. Then, for any* $p > 0$ *and the corresponding* $\lambda^*(p)$*, there exists at least one optimal solution* x^* *of problem* (20) *such that* x^* *is a primal feasible solution of problem* (1).

Suppose that the optimal solutions of (20) corresponding to $\lambda^*(p)$ are all primal infeasible. Then, we have

$$\eta = \min_{x \in S} L_p(g(x), f(x), \lambda^*(p)) - \min_{x \in X \setminus S} L_p(g(x), f(x), \lambda^*(p)) > 0. \tag{36}$$

For any $x \in S$, by the continuity of $L_p(g(x), f(x), \lambda)$, there exists an $\varepsilon_1 > 0$ such that for any $0 \leqslant \varepsilon \leqslant \varepsilon_1$,

$$L_p(g(x), f(x), \lambda^*(p) + \varepsilon) > L_p(g(x), f(x), \lambda^*(p)) - \frac{\eta}{2},$$

which implies

$$\min_{x \in S} L_p(g(x), f(x), \lambda^*(p) + \varepsilon) > \min_{x \in S} L_p(g(x), f(x), \lambda^*(p)) - \frac{\eta}{2}. \tag{37}$$

Similarly, there exists an $\varepsilon_2 > 0$ such that for any $0 \leqslant \varepsilon \leqslant \varepsilon_2$,

$$\min_{x \in X \setminus S} L_p(g(x), f(x), \lambda^*(p) + \varepsilon) < \min_{x \in X \setminus S} L_p(g(x), f(x), \lambda^*(p)) + \frac{\eta}{2}. \tag{38}$$

Notice that

$$\min_{x \in S} L_p(g(x), f(x), \lambda^*(p)) - \frac{\eta}{2} = \min_{x \in X \setminus S} L_p(g(x), f(x), \lambda^*(p)) + \frac{\eta}{2}. \tag{39}$$

Choose an ε satisfying $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$. Then we have from (37), (38) and (39) that

$$\min_{x \in S} L_p(g(x), f(x), \lambda^*(p) + \varepsilon) > \min_{x \in X \setminus S} L_p(g(x), f(x), \lambda^*(p) + \varepsilon).$$

Since $L_p(g(x), f(x), \lambda)$ is regular, for $x \in X \setminus S$, we have

$$L_p(g(x), f(x), \lambda^*(p) + \varepsilon) > L_p(g(x), f(x), \lambda^*(p)).$$

Thus,

$$\begin{aligned}\phi_p(\lambda^*(p)+\varepsilon) &= \min_{x\in X} L_p(g(x),f(x),\lambda^*(p)+\varepsilon)\\ &= \min_{x\in X\setminus S} L_p(g(x),f(x),\lambda^*(p)+\varepsilon)\\ &> \min_{x\in X\setminus S} L_p(g(x),f(x),\lambda^*(p))\\ &= \min_{x\in X} L_p(g(x),f(x),\lambda^*(p))\\ &= \phi_p(\lambda^*(p)).\end{aligned}$$

This is a contradiction to the optimality of $\lambda^*(p)$ in problem (21).

LEMMA 5. *Suppose that $L_p(g(x),f(x),\lambda)$ is a regular GLF. For any $p>0$,* if $\lambda>\lambda^*(p)$, *then any optimal solution of* (20) *corresponding to λ is primal feasible for problem* (1).

From Lemma 4, there must exist an optimal solution of (20) with $\lambda=\lambda^*(p)$, x^*, that is primal feasible. Since $g(x^*)\leqslant 0$, for $\lambda>\lambda^*(p)$, we have

$$L_p(g(x^*),f(x^*),\lambda)\leqslant L_p(g(x^*),f(x^*),\lambda^*(p)). \tag{40}$$

Since L_p is regular, for any $x\in X\setminus S$ and $\lambda>\lambda^*(p)$, we have

$$L_p(g(x^*),f(x^*),\lambda^*(p))\leqslant L_p(g(x),f(x),\lambda^*(p)) < L_p(g(x),f(x),\lambda) \tag{41}$$

Combining (40) and (41), we obtain

$$L_p(g(x^*),f(x^*),\lambda) < L_p(g(x),f(x),\lambda),\quad \forall x\in X\setminus S.$$

Thus, any optimal solution of (20) must be primal feasible, when $\lambda>\lambda^*(p)$.

THEOREM 3. *Suppose that the GLF $L_p(g(x),f(x),\lambda)$ is regular. For sufficiently large p and $\lambda>\lambda^*(p)$, any optimal solution of problem (20) is a primal optimal solution of problem (1).*

This conclusion can be obtained directly from Lemmas 2 and 5.

6. Illustrative examples

In this section, we present two examples to illustrate the utilization of the GLF. We first consider Example 5.12 in [7] .

EXAMPLE 5.

$$\begin{aligned}&\min\quad f(x)=3x_1+2x_2\\ &\text{s.t}\quad g_1(x)=10-5x_1-2x_2\leqslant 7,\\ &\qquad g_2(x)=15-2x_1-5x_2\leqslant 12,\\ &\qquad x\in X=\{x: 0\leqslant x_1\leqslant 1,\ 0\leqslant x_2\leqslant 2, 8x_1+8x_2\geqslant 1, x \text{ integer}\}.\end{aligned} \tag{42}$$

Note that g_i, $i = 1, 2$, are constructed in such a way to ensure both g_i, $i = 1, 2$, are strictly positive for any $x\in X$. The above example problem can be converted into the following equivalent singly-constrained problem using the p-norm surrogate constraint method [4] with $p = 9$,

$$\begin{aligned} \min \quad & f(x) = 3x_1 + 2x_2 \\ \text{s.t.} \quad & g(x) = (\mu_1[g_1(x)]^9 + \mu_2[g_2(x)]^9)^{1/9} - 4.775 \leqslant 0, \\ & x \in X = \{x : 0 \leqslant x_1 \leqslant 1,\ 0 \leqslant x_2 \leqslant 2, 8x_1 + 8x_2 \geqslant 1,\ x \quad \text{integer}\}, \end{aligned} \tag{43}$$

where $\mu = [0.016, 0.00013]$. We now construct a regular GLF for (43) as follows.

$$L_p(g(x), f(x), \lambda) = F_p[S(f(x)) + T(g(x))]. \tag{44}$$

If we take $F_p(x) = x^{1/p}$, $S(f(x)) = f(x)^p$ and $T(g(x)) = \exp(p\lambda g(x))$ in (44), we get the nonlinear Lagrangian formula as in Example 4,

$$L_p(g(x), f(x), \lambda) = [f(x)^p + \exp(p\lambda g(x))]^{1/p}. \tag{45}$$

Applying formula (45) to Example 5, we obtain a relaxation problem of (42) as follows.

$$\begin{aligned} & \min \{(3x_1 + 2x_2)^p + \exp[p\lambda((\mu_1[g_1(x)]^9 + \mu_2[g_2(x)]^9)^{1/9} - 4.775)]\}^{1/p} \\ & \text{s.t.} \quad x \in X = \{x : 0 \leqslant x_1 \leqslant 1,\ 0 \leqslant x_2 \leqslant 2, 8x_1 + 8x_2 \geqslant 1,\ x \text{ integer}\}. \end{aligned} \tag{46}$$

The dual function in Figure 1 demonstrates a unimodality of the dual function with respect to λ, although the magnitude of the negative slope on the right of the maximum point is very small. For $p \geqslant 8$, we can solve the example problem by solving (46) for any $\lambda \geqslant 52$ by a branch and bound procedure. The algorithm identifies the optimal solution $x^* = (0, 2)$ with $f(x^*) = 4$.

If take $F_p(x) = \frac{1}{p}\ln(x)$, $S(f(x)) = \exp(pf(x))$ and $T(g(x)) = \exp(p\lambda g(x))$ in (44), we get the nonlinear Lagrangian formula as in Example 1,

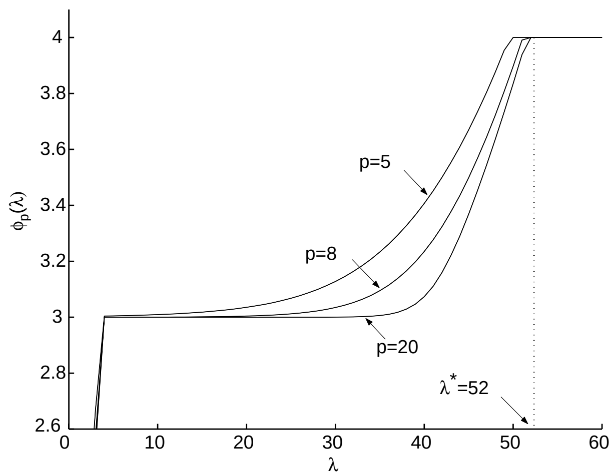

Figure 1. Picture of $\phi_p(\lambda) = \min_{x \in X}[f(x)^p + \exp(p\lambda g(x))]^{1/p}$.

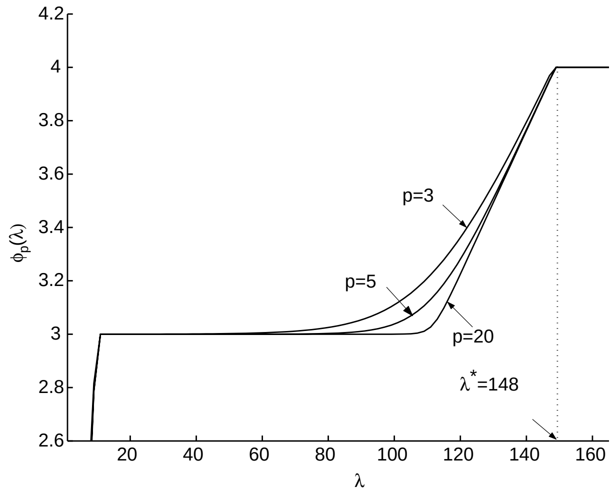

Figure 2. Picture of $\phi_p(\lambda) = \min_{x\in X}\frac{1}{p}\ln[\exp(pf(x)) + \exp(p\lambda g(x))]$.

$$L_p(g(x), f(x), \lambda) = \frac{1}{p}\ln[\exp(pf(x)) + \exp(p\lambda g(x))]. \tag{47}$$

Applying (47) to Example 5, we have

$$\begin{aligned} \min \quad & \frac{1}{p}\ln\{\exp[p(3x_1 + 2x_2)] + \exp[p\lambda((\mu_1[g_1(x)]^9 \\ & + \mu_2[g_2(x)^9)]^{1/9} - 4.775)]\} \\ \text{s.t.} \quad & x \in X = \{x : 0 \leqslant x_1 \leqslant 1,\ 0 \leqslant x_2 \leqslant 2, 8x_1 + 8x_2 \geqslant 1,\ x \text{ integer}\}. \end{aligned} \tag{48}$$

The dual function in Figure 2 demonstrates a unimodality of the dual function with respect to λ, although the magnitude of the negative slope on the right of the maximum point is very small. For any $p \geqslant 5$, we can solve the example problem by solving (48) for any $\lambda \geqslant 148$ by a branch and bound procedure. The algorithm identifies the optimal solution $x^* = (0, 2)$ with $f(x^*) = 4$.

If we take $F_p(x) = x$, $S(f(x)) = f(x)$ and $T(g(x)) = (1/\lambda)\exp(\lambda g(x))$, where $\lambda \geqslant p > 0$, in (44), we get the nonlinear Lagrangian formula as in Example 2,

$$L_p(g(x), f(x), \lambda) = f(x) + \frac{1}{\lambda}\exp(\lambda g(x)),\ \lambda \geqslant p > 0. \tag{49}$$

Applying (49) to Example 5, we have

$$\begin{aligned} \min \quad & \{(3x_1 + 2x_2) + \frac{1}{\lambda}\exp[\lambda((\mu_1[g_1(x)]^9 + \mu_2[g_2(x)]^9)^{1/9} - 4.775)]\} \\ \text{s.t.} \quad & x \in X = \{x : 0 \leqslant x_1 \leqslant 1,\ 0 \leqslant x_2 \leqslant 2, 8x_1 + 8x_2 \geqslant 1,\ x \text{ integer}\}. \end{aligned} \tag{50}$$

The dual function in Figure 3 demonstrates a unimodality of the dual function with respect to λ. For any $p \geqslant 1.5$, we can solve the example

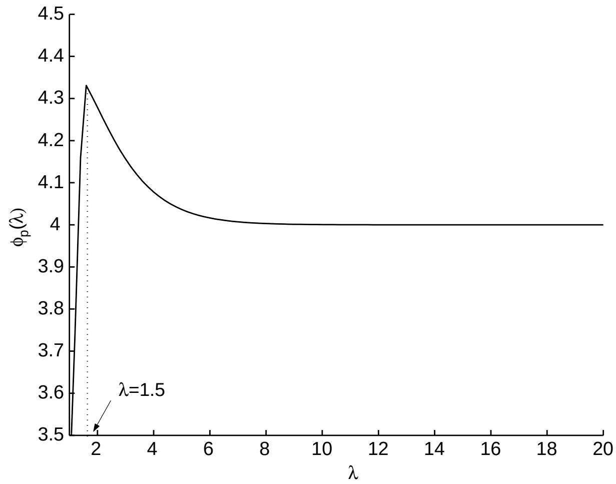

Figure 3. Picture of $\phi_p(\lambda) = \min_{x \in X}\{f(x) + \frac{1}{\lambda}\exp(\lambda g(x))\}$.

problem by solving (50) for any $\lambda \geqslant p$ by a branch and bound procedure. The algorithm identifies the optimal solution $x^* = (0, 2)$ with $f(x^*) = 4$.

EXAMPLE 6. Consider a redundancy optimization problem in a network system consisting of n subsystems. The reliability of the ith subsystem is $R_i(x_i) = 1 - (1 - r_i)^{x_i}$, where x_i is the number of the same components in parallel in the ith subsystem and $r_i \in (0, 1)$ is the given reliability of the component in the ith subsystem. Also, denote by C(x) the total resource consumed when adopting decision x. Consider an instance of this reliability optimization problem with five elements and a single constraint in [12].

$$\begin{aligned} \min \quad & Q(x) = 1 - R_1R_2 - (1 - R_2)R_3R_4 - (1 - R_1)R_2R_3R_4 \\ & \qquad - R_1(1 - R_2)(1 - R_3)R_4R_5 - (1 - R_1)R_2R_3(1 - R_4)R_5 \\ \text{s.t.} \quad & C(x) = x_1x_2 + 3x_2x_3 + 3x_2x_4 + x_1x_5 \leqslant 28, \\ & 1 \leqslant x_i \leqslant 6, \quad x \quad \text{integer}, \quad i = 1, ..., 5. \end{aligned} \tag{51}$$

where $r_1 = 0.7, r_2 = 0.85, r_3 = 0.75, r_4 = 0.8, r_5 = 0.9$. Applying formula (45) to (51) with $p \geqslant 2$, we can solve Example 6 for any $\lambda \geqslant 8$. Applying formula (47) to (51) with $p \geqslant 3$, we can solve this problem for any $\lambda \geqslant 3$. And applying formula (49) to (51) with any $p \geqslant 6.5$, we can get the optimal solution for any $\lambda \geqslant p$. These three algorithms all identify the optimal solution $x^* = (2, 1, 4, 4, 1)$ with $Q(x^*) = 0.000656$.

As witnessed from above examples, adoption of the proposed GLF transfers an integer programming problem with nonlinear constraints into an equivalent integer programming problem with box constraints that is easier to be solved than the original problem. Note that for a p and a λ

that are sufficiently large, no dual search is needed. Thus, only one resulting nonlinear Lagrangian problem needs to be solved by a branch and bound method.

7. Conclusion

The research output presented in the paper can be regarded as a unified framework of nonlinear Lagrangian formulations that have been developed in order to pinpoint the optimal solution of the primal problem via dual search. Specifically, a general form of nonlinear Lagrangian functions has been identified with which the dual search is guaranteed to generate an optimal solution of the primal problem.

Acknowledgements

This research was partially supported by the Research Grants Council of Hong Kong, Grant CUHK 4214/01E.

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